Sections 6.1, 6.2*-6.4*, 6.5

Atsuya Kumano
Penn State University

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6.1: Inverse functions

An **invertible function** is a function such that its effect can be reversed (p.385 Figure 5). A geometric way to check whether a function is invertible or not is to use the **horizontal line test** (p.384). An algebraic way to check is to take the derivative. If $f'(x) > 0$ or $f'(x) < 0$ for all x, then f is one-to-one.

Read Example 1 and 2. Then consider the following problem.

Exercise 1. Is $f(x) = (x-1)(x-3)^2$ invertible?

Solution: Draw the graph (by hand or go to WolframAlpha). You find that the horizontal line $y = 0$ intersects the graph twice. Therefore, f is not invertible.

Read Definition 2. Pay attention to the domains and ranges of f and f^{-1}. We will discuss why it is important to note where the range of domain of the inverse function f^{-1} in the next section.

Read Page 386 carefully. $\boxed{3}$ and $\boxed{4}$ are just reformulations of the definition of invertible functions (i.e. that f^{-1} reverses the effect of f).

$\boxed{5}$ discusses a way to find the inverse function f^{-1}.

Exercise 2. Find the inverse function of $f(x) = \frac{1-\sqrt{x}}{1+\sqrt{x}}$.

Solution: Set $y = \frac{1-\sqrt{x}}{1+\sqrt{x}}$. Our goal is to solve for x. Multiply both sides by $1+\sqrt{x}$ to get

$$(1+\sqrt{x})y = 1-\sqrt{x}.$$

Then, move the terms around

$$\begin{aligned} &y + y\sqrt{x} = 1-\sqrt{x} \\ \iff &(1+y)\sqrt{x} = 1-y \\ \iff &\sqrt{x} = \frac{1-y}{1+y}. \end{aligned}$$

Square both sides to conclude that $x = \left(\frac{1-y}{1+y}\right)^2$. So $f^{-1}(x) = \left(\frac{1-x}{1+x}\right)^2$.

Exercise 3. Check the solution of the previous exercise. In other words, show that $f(f^{-1}(x)) = x = f^{-1}(f(x))$.

Theorem 7 is a quite important one. It says that you can compute the derivative of f^{-1} indirectly, only using f' and f^{-1}. Take a look at Example 7. The problem asks to find $(f^{-1})'(1)$ for $f(x) = 2x+\cos x$. The first thing that one tries (at least I do) is to find the explicit formula of f^{-1}, then differentiate f^{-1}. So set $y = 2x+\cos x$, then try to solve for x...But how? It turns out there isn't an easy way to find the inverse of this function (input this into WolframAlpha see what it returns: "inverse of 2x + cos(x)"). Now, Theorem 7 comes to rescue, which tells you that $(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$. We know that $f'(x) = 2-\sin(x)$, so we can reduce the last expression to $\frac{1}{2-\sin(f^{-1}(1))}$. The last missing piece is the value of $f^{-1}(1)$. We don't know the formula of f^{-1}, so it might seem that we're stuck again. However, there's a way to work around this issue. Set $y = f^{-1}(1)$ for the sake of notation. By definition, y is the value such that $f(y) = 1$. With a little ingenuity, we see that $y = 0$ works: $f(0) = 0 + cos0 = 1$. So we conclude that $y = 0$. (Since f is one-to-one, we know that there is only one such value.) Therefore, $0 = y = f^{-1}(1)$, and the solution is $\frac{1}{2-\sin(f^{-1}(1))} = \frac{1}{2-\sin 0} = \frac{1}{2}$.

Problems

1. (Exam 1 Sample A) Let $f(x) = 4x + \cos \pi x$. Find $(f^{-1})'(3)$.

2. (Exam 1 Sample C) Suppose that f^{-1} is the inverse function of a differentiable function f such that $f(2) = 5$ and $f'(2) = \frac{1}{3}$. Find $(f^{-1})'(5)$.

6.2*: The Natural Logarithmic Function

In the remaining sections of Chapter 6, we discuss several functions that we will use heavily in the rest of the course. The first is ln, the natural log function. I hope you are familiar with this function already. $\boxed{2}$ and $\boxed{3}$ list important properties of ln.

Exercise 4. Work out Example 2 without looking at the solution.

It is useful to have the graph of ln in mind. Input "Plot[Ln(x), (x, 0, 10)]" in WolframAlpha. This will give you the plot of $\ln(x)$ from $x = 0$ to 10. Things that you should note are:

1. ln is **not defined** for $x \leq 0$.

2. ln is one-to-one (for $x > 0$), hence invertible.

3. $\ln(x)$ is negative for $0 < x < 1$, and it goes to $-\infty$ as $x \to 0$ ($\boxed{4}$).

4. $\ln(x)$ is zero when $x = 1$.

5. $\ln(x)$ is positive for $x > 1$, and it goes to $+\infty$ as $x \to \infty$ ($\boxed{4}$).

You should become comfortable with differentiating ln using the chain rule. $\boxed{2}$ says $(\ln(x))' = \frac{1}{x}$, and if we combine $\boxed{2}$ with the chain rule, we get $\boxed{6}$. $\boxed{6}$ gives you two formulas, but personally I like this one better: $\frac{d}{dx}(\ln g(x)) = \frac{g'(x)}{g(x)}$.

Exercise 5. Find $(\ln(2x))'$.

A common mistake is to forget using the chain rule, and do $(\ln(2x))' = \frac{1}{2x}$. The correct solution is the following.

Solution: Set $g(x) = 2x$. By the chain rule, $(\ln(2x))' = (\ln g(x))' = \frac{g'(x)}{g(x)} = \frac{2}{2x} = \frac{1}{x}$.

You should read Examples 6, 7, and 8 carefully.

Another property of ln is worth noting:

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}; \qquad \int \frac{1}{x} dx = \ln|x| + C$$

($\boxed{7}$ and $\boxed{8}$). Read Examples 11, 12, and 13 carefully. We will go through Example 13 in class. The result of Example 13 is worth memorizing: $\int \tan x dx = \ln|\sec x| + C$. This formula (the integral of tangent) will come handy in some of the more difficult integration problems.

The last topic in this section is logarithmic differentiation. Take a look at the expression in Example 14: $y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$. If you try to compute y' in a straightforward fashion, it would involve lots of keeping track of product rules and chain rules, so you'd rather not do that. Notice that y is a product of the three expressions: $x^{3/4}$, $\sqrt{x^2+1} = (x^2+1)^{1/2}$, and $(3x+2)^5$. In such case, logarithmic differentiation makes computing y' much, much easier. The main idea is that differentiating $\ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$ involves much less efforts than y. You should work through this example without looking at the solution.

Problems

1. Expand the quantity $\ln\sqrt{\frac{x^2}{z^3}}$.

2. Expand $\ln \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

3. Evaluate $\lim_{x\to 1^+} \ln \frac{1}{x-1}$.

4. Differentiate $\ln(x^3 + \sqrt{x^2 - 1})$.

5. $\int \frac{dt}{8-3t}$.

6. $\int \frac{\cos x}{2+\sin x} dx$.

7. $\int \frac{(\ln x)^2}{x} dx$.

8. (Exam 1 Sample A) Given that $a, b > 0$ and $a \neq b$, evaluate $\lim_{x\to\infty} \ln(3+ax) - \ln(2+bx)$.

9. (Exam 1 Sample A) Evaluate the integral $\int_e^{e^2} \frac{2\ln x}{x} dx$.

10. (Exam 1 Sample B) If $f(x) = \ln(x^2 \sin x)$ find $f'(x)$.

11. (Exam 1 Sample C) Differentiate the function $f(x) = \ln(\sin(\ln x))$.

12. Differentiate the function $y = e^{5\cos\sqrt{x}}$.

6.3*: The Natural Exponential Function

The exponential function, exp, is also something that you should be already familiar with. The textbook tells you to think of exp as the inverse of ln ($\boxed{2}$), but you can also think of it as the function $\exp(x) = e^x$, where e is Euler's constant. The properties of exp are:

1. exp is one-to-one, hence invertible (just like ln).

2. $\exp(\ln x) = x$, $\ln(\exp x) = x$ i.e. exp and ln are inverses of each other ($\boxed{2}$). We can rewrite this as $e^{\ln x} = x$ and $\ln(e^x) = x$ also ($\boxed{4}$ and $\boxed{5}$).

3. The range of exp is $(0, \infty)$, i.e. $\exp(x)$ is always positive.

4. $e^{x+y} = e^x e^y$; $e^{x-y} = \frac{e^x}{e^y}$; $(e^x)^y = e^{xy}$.

Just like ln, it's useful to remember what the graph of exp looks like. Use WolframAlpha to see the graph (try "exp(x)").

1. $\lim_{x\to\infty} e^x = \infty$ i.e. e^x is big when x is a large positive number.

2. $\lim_{x\to-\infty} e^x = 0$, i.e. e^x is close to zero when x is a large negative number.

3. $e^0 = 1$. Compare this with $\ln 1 = 0$.

Two lectures ago, I made a small fuss about the range and domain of the inverse function. This is a good moment to discuss it. Go back to Definition $\boxed{2}$ in p385. Take $f(x) = \exp x$. Then, $f^{-1}(x) = \ln x$, and $f^{-1}(f(x)) = x$ for all x. Note that it does **not** make sense to write $f^{-1}(-1) = \ln(-1)$, because $\ln(x)$ is **not defined** at $x = -1$ or any negative x. So, if $f(x)$ were negative for some x, then we'd in a big trouble, because $f^{-1}(f(x))$ is not defined. But we don't get into such trouble at all. Definition $\boxed{2}$ says the domain of f^{-1} is the same as the range of $f = \exp$, which is $(0, \infty)$, all positive numbers.

Another good example to consider is $\tan x$. $\tan x$ is **not** one-to-one over the entire x-axis (it's a periodic function). However, it is one-to-one when restricted to $(-\pi/2, \pi/2)$, hence invertible. We call the inverse of tan as arctan. The domain and range of tan are $(-\pi/2, \pi/2)$ and $(-\infty, \infty)$, respectively. So the domain and range of arctan are $(-\infty, \infty)$ and $(-\pi/2, \pi/2)$.

This might seem like a technicality that only mathematicians care about. (In fact, it kind of is.) But sometimes this can become the source of all problems when doing integrations. Just keep in mind that mindlessly taking inverses of functions can cause troubles. When something is wrong, think once again whether your function is invertible, and, if it is, what a domain of the inverse is.

Differential and integral properties of exp are quite simple:

$$\frac{d}{dx}e^x = e^x; \qquad \int e^x dx = e^x + C.$$

You just need to be able to use them well in conjunction with the chain rule and product rule.

Exercise 6. Find $\frac{d}{dx}e^{x^2}$.

Solution: By the chain rule, $\frac{d}{dx}e^{x^2} = (x^2)'e^{x^2} = 2xe^{x^2}$.

Exercise 7. n is an integer. Find $\frac{d}{dx}e^{x^n}$.

Problems

1. Find $(e^{2x})'$.
2. Find $\int e^{4x}dx$.
3. Find $(e^{x^2})'$.
4. Evaluate $\lim_{x\to\infty} e^{1/x^2}$.
5. Evaluate $\lim_{x\to\infty} \frac{x^3}{e^x}$.
6. Find $(\sin(e^t) + e^{\sin t})'$.
7. $\int e^x(4+e^x)^5dx$.
8. $\int e^x\sqrt{1+e^x}dx$.
9. $\int e^{\tan x}\sec^2 xdx$.
10. (Exam 1 Sample C) Find the inverse function of f, $f^{-1}(x)$, if $f(x) = \frac{e^x}{1+e^x}$.
11. (Exam 1 Sample D) Find the derivative of $y = \sin^{-1}(e^{-x})$.
12. Find $\frac{d}{dx}e^{x\ln a}$ (a is a real number).

6.4*: General Logarithmic and Exponential Functions

We discussed the exponential function with base e in the previous section. There are cases where we want to take the exponent of numbers other than e, then manipulate, differentiate, and/or integrate them. The definition of a general exponential function is $a^x = e^{x\ln a}$. Note that if $a = e$, then $e^{x\ln e} = e^x$. All the usual properties of exp function apply to general exponential functions ($\boxed{3}$). $\frac{d}{dx}(a^x) = a^x \ln a$ ($\boxed{4}$) and $\int a^x dx = \frac{a^x}{\ln a} + C$ (p440) are formulas that you should memorize.

Exercise 8. Differentiate 5^x.

Solution: By the formula, $(5^x)' = \ln 5 \cdot 5^x$.

Exercise 9. Differentiate $f(x) = 3^{\cos 2x}$.

Solution: By the chain rule, $f'(x) = \ln 3 \cdot 3^{\cos 2x} \cdot (\cos 2x)' = -2\ln 3 \cdot 3^{\cos 2x} \cdot \sin 2x$.

There is a nice discussion of how know when to use the Power Rule or Exponential Rule on p441, which you should read carefully.

Now that we know what general exponential functions are, we can also define general log functions. The textbook's definition is that the log function with base a is the inverse of a^x ($\boxed{5}$). But it's easier to think of it simply as $\log_a x = \frac{\ln x}{\ln a}$ ($\boxed{6}$). The consequence of this formula is that $\frac{d}{dx}\log_a x = \frac{1}{x\ln a}$.

Problems

1. Find $(x^4 + 5^x)'$.
2. Evaluate $\int x^4 + 5^x dx$.
3. Evaluate $\int x2^{x^2} dx$.
4. Find $(x^{\cos x})'$.

5. $\int \frac{\log_{10} x}{x} dx$.

6. $\int \frac{2^x}{2^x+1} dx$.

7. (Exam 1 Sample B) If $f(x) = 3^x$, find the second derivative $f''(x)$.

8. (Exam 1 Sample B) If $f(x) = x^{2x}$, find $f'(e)$.

9. (Exam 1 Sample C) Differentiate $y = 3^{x^2}$.

6.6: Inverse Trigonometric Functions

As discussed in Tuesday's lecture, the trig functions are only invertible on certain domains. Pay attention to what the domains of arcsin, arccos, and arctan are. You should also know the formulas for the derivatives of inverse trigonometric functions, which is summarized in table 11. The derivations of these formula, which are scattered throughout the section, utilize implicit differentiation.

Problems

1. Simplify $\tan(\sin^{-1} x)$.

2. Show that $(\sec^{-1} x)' = \frac{1}{x\sqrt{x^2-1}}$.

3. $(\arcsin \sqrt{\sin \theta})'$.

4. $(\arctan \sqrt{\frac{1-x}{1+x}})'$.

5. $\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} dx$.

6. $\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$.

Sections 7.1-7.5

Atsuya Kumano
Penn State University

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7.1: Integration by Parts

The main reason why integration by parts is useful is condensed in Examples 1-3. This technique is useful in many situations, and, in my opinion, the most important one that we cover in this course. Example 4 shows a neat trick involving integration by parts. You should remember this trick, because if you didn't know, you wouldn't think about it.

Another way of writing down the formula for integration by parts is this:

$$\int f(x)g(x)dx = F(x)g(x) - \int F(x)g'(x)dx,$$

where $F(x) = \int f(x)dx$ (omitting the integration constant C). I like this way of memorizing it better than the one in the textbook.

Problems

1. $\int x \cos x dx$.

2. $\int e^x \sin 2x dx$.

3. $\int \ln x dx$.

4. $\int (\ln x)^2 dx$.

5. (More Integration Practice #25) $\int \sec^{-1} x dx$ (assume $x > 1$).

6. (Sample A #14) $\int x^2 \ln x dx$.

7. $\int t^3 e^{-t^2} dx$

7.2: Trigonometric Integrals

For one reason or another (one of the major culprits being "Fourier series), you will end up doing a lot of integration involving sines and cosines later on in your academic career if you're in engineering, physics, or anything related to them. This section presents some nifty tricks to deal with integrals of that kind. The formulas that you need to memorize are the following:

$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1 \\
2 \sin x \cos x &= \sin 2x \\
1 + \tan^x &= \sec^2 x \\
\sin^2 x &= \frac{1 - \cos 2x}{2} \\
\cos^2 x &= \frac{1 + \cos 2x}{2} \\
\sin x \cos y &= \frac{\sin(x-y) + \sin(x+y)}{2} \\
\sin x \sin y &= \frac{\cos(x-y) - \cos(x+y)}{2} \\
\cos x \cos y &= \frac{\cos(x-y) + \cos(x+y)}{2}
\end{aligned}$$

Problems

1. $\int \sin^4 x \cos^3 x dx$.

2. $\int \cos^2 x \sin 2x dx$.

3. $\int \sin^2 x \cos^2 x dx$.

4. $\int \tan^5 x \sec^3 x dx$.

5. $\int \tan^2 x \sec^4 x dx$.

6. $\int \sin 4\theta \cos 3\theta dx$.

7.3: Trigonometric Substitution

This section is similar in spirit with 7.2: there are certain patterns that you should recognize, and when you do, there are certain change of variables that you should perform. The table in p502 gives three patterns, to which you respond by setting your variable to a trig function. You should memorize all three.

Problems

1. $\int \sqrt{1-x^2} dx$

2. $\int x^3 \sqrt{1-x^2} dx$

3. $\int \frac{dx}{t^2\sqrt{t^2-4}}$

4. $\int \frac{dx}{t^2-6t+13}$.

5. (More Integration Practice #22) $\int \frac{dx}{\sqrt{e^{2x}-1}}$.

6. (Integration Practice #20) $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$.

7. (Sample #17) $\int \frac{1}{x^4\sqrt{x^2-9}} dx$.

7.4: Integration of Rational Functions by Partial Fractions

This method is easier to learn than integration by parts or change of variables. The main idea is to reduce integration problems of a rational function (i.e. products and quotients of polynomials) to integrating $\frac{1}{x}$ and $\frac{1}{x^2+1}$, which we know very well. There are four cases that the section presents. Read each of them, and get your hands dirty afterwards.

Problems

1. $\int \frac{x}{x-1} dx$.
2. $\frac{x+4}{x^2+2x+5} dx$
3. $\int \frac{x}{x^2-4} dx$
4. $\int \frac{x^6}{x^2-4} dx$
5. $\int \frac{x^2+1}{(x-3)(x-2)^2} dx$.
6. $\int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)} dx$.
7. $\int \frac{x^4+3x^2+1}{x^5+5x^3+5x} dx$.
8. (Sample A #12) $\int \frac{5x+2}{x^2+x} dx$.
9. (Sample C #14) $\int \frac{x+1}{x^2-5x+6} dx$.

7.5: Strategy for Integration

Now that you have several integration techniques up your sleeve, you are ready to start doing *problem solving*. There's a book about problem solving by a mathematician George Polya called "How to Solve It." `https://en.wikipedia.org/wiki/How_to_Solve_It` Its wikipedia page describes the "four principles" of problem solving, which is the essense of the book. You should read it (not the book, but the wiki page).

In my opinion, reading this book doesn't make you a better problem solver. The four principles are so obvious that no one has a hard time understanding them. (I especially like one of the suggestions for the second principle: "Be creative.") The tough part, however, is to actually do it. There's no algorithm for hard problems, because every hard problem is hard in its own way. The only way of become a good problem-solver is by experience. So you can skip the reading part of this section (though you might want to come back to it every now and then), and skip to p523-524, which lists 82 integration problems. Do as many of them as you can.

Problems

1.