Sections 6.8 & 7.8

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6.8: Indeterminate Forms

As per usual, you should take a look at all examples. There are seven types of indeterminate forms discussed in this section:

1. $\frac{0}{0}$
2. $\frac{\infty}{\infty}$
3. $0 \cdot \infty$
4. $\infty - \infty$
5. 0^0
6. ∞^0
7. 1^∞

Make sure you read examples corresponding to each of these.

Roughly, l'Hopital's rule says this: if $f(a) = g(a) = 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\text{slope of } f(x) \text{ at } x=a}{\text{slope of } g(x) \text{ at } x=a}$. As we know, "slope of $f(x)$ at $x = a$" is $f'(a)$. So we get the formula $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$. You can use l'Hopital's when the limit has the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Other types of indeterminate limits can be computed by applying l'Hospital.

Exercise 1. Show that $\lim_{x\to 0^+} x\ln x = 0$.

<u>*Solution*</u>: Note that $f(x) = x\ln x$ is not defined when $x \le 0$. But the graph of $f(x)$ (Figure 5 in p473) approaches zero as x get closer and closer to zero. We have $\lim_{x\to 0^+} x = 0$ and $\lim_{x\to 0^+} \ln x = -\infty$, so $\lim_{x\to 0^+} x\ln x$ has the indeterminate form $0\cdot(-\infty)$. $x\ln x$ is not a fraction, but it turns out there's a clever application of l'Hopital's rule. First write $x\ln x = \frac{\ln x}{1/x}$. Then, apply l'Hopital's rule to get $\lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0$.

The following is an example (taken from Wikipedia) where you cannot apply l'Hopital.

Exercise 2. Find $\lim_{x\to\infty} \frac{x+\sin x}{x}$.

<u>*Solution*</u>: (Wrong solution) By l'Hopital, $\lim_{x\to\infty} \frac{x+\sin x}{x} = \lim_{x\to\infty} \frac{1+\cos x}{1}$. This limit does not exist, because $\cos x$ diverges.

This solution is incorrect, because the limit actually does exist! $\lim_{x\to\infty} \frac{x+\sin x}{x} = \lim_{x\to\infty} 1 + \frac{\sin x}{x} = 1$.

Why does l'Hopital fail in this example? First, lets take a look at the conditions that l'Hopital holds (also taken from Wikipedia).

1. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\pm\infty$
2. $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists
3. $g'(x) \neq 0$ for each $x \neq a$ in some open interval containing a.

In this course, you can take a cavalier attitude in applying l'Hopital, since we usually deal with "nice" functions with nice limits. But just remember that you can't apply l'Hopital to any functions. When you are getting strange results, you might want to consider whether your function satisfies the conditions above.

Another example in which an application of l'Hopital can go wrong is Example 5 in p472: $\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x}$.

The last three types of indeterminate forms, namely 0^0, ∞^0, and 1^∞, can be solved using the same principle: when you see something complicated involving powers, use log (just like how you find the derivative of things like

x^x with implicit differentiation). Or you could use the definition of general powers, as Example 10 demonstrates.

Problems

1. $(\frac{0}{0})$ $\lim_{x\to\infty} \frac{e^{-x}}{x^2}$
2. $(\frac{\infty}{\infty})$ $\lim_{x\to\infty} \frac{\ln\sqrt{x}}{x^2}$
3. $(0\cdot\infty)$ $\lim_{x\to 0^+} x\ln x$
4. $\lim_{x\to\infty} x\sin(\pi/x)$
5. $(\infty-\infty)$ $\lim_{x\to 1} \frac{x}{x-1} - \frac{1}{\ln x}$
6. $\lim_{x\to 0} \csc x - \cot x$
7. (0^0) $\lim_{x\to 0^+} x^{\sqrt{x}}$
8. (∞^0) $\lim_{x\to\infty} x^{\frac{1}{\ln x}}$
9. (1^∞) $\lim_{x\to 1}(2-x)^{\tan(\pi x/2)}$
10. Exercises 1-66 (p477-478)

7.8: Improper Integrals

What's so "improper" about improper integrals? In a formal and rigorous definition of integration of single-variable functions, an integral is defined only over intervals of *finite* lengths. (A bit of digression here: Newton and Leipniz invented integration, but it was Bernhard Riemann who made the theory of integration logically sound. The integration that we use in calculus courses is called "Riemann integration," in order to distinguish it from other definitions of integrations.) This is why an integral like $\int_0^\infty e^{-x}dx$ is "improper."

Improper integrals of this type are those of the forms $\int_a^\infty f(x)dx$, $\int_{-\infty}^a f(x)dx$, and $\int_{-\infty}^\infty f(x)dx$. For practical purposes, you already know how to do these kinds of integrals (i.e. do the corresponding indifinite integral, then plug in

∞ (or $-\infty$). However, it's sometimes useful to know that when you "plug in ∞," what you actually do is, by definition ($\boxed{1}$), to take your variable to the limit $x \to \infty$. In Example 2 (p545), you can use l'Hopital's rule when evaluating te^t at $t = \infty$, because of the definition.

Exercise 3. Determine whether the integral $\int_{-\infty}^{\infty} x^3 dx$ is convergent.

Solution: (Wrong solution)

$$\int_{-\infty}^{\infty} x^3 dx = \int_0^{\infty} x^3 dx + \int_{-\infty}^{0} x^3 dx = \lim_{t\to\infty} \int_0^t x^3 dx + \lim_{t\to-\infty} \int_t^0 x^3 dx$$
$$= \lim_{t\to\infty} \frac{t^4}{4} - \lim_{t\to-\infty} \frac{t^4}{4} = \lim_{t\to\infty} \frac{t^4}{4} - \lim_{t\to\infty} \frac{t^4}{4} = \lim_{t\to\infty} \left(\frac{t^4}{4} - \frac{t^4}{4}\right) = 0.$$

(Correct solution)

$$\int_0^{\infty} x^3 dx = \lim_{t\to\infty} \int_0^t x^3 dx = \lim_{t\to\infty} \frac{t^4}{4}$$

So $\int_0^{\infty} x^3 dx$ diverges. **If one of the integrals diverges, then we say that the original integral diverges** (it's how we defined this type of improper integral in $\boxed{1}$(c)). Hence, $\int_{-\infty}^{\infty} x^3 dx$ diverges.

Note that the definition is similar for an integral with discontinuity in the region of integration. See $\boxed{3}$(c), then read Example 7 to get the feel of it.

Example 4 and $\boxed{2}$ is worth remembering. Perhaps not now, but we'll use this fact in Chapter 11.

Improper integrals of the second type are those with discontinuities. Read $\boxed{3}$, then take a look at the following example.

Exercise 4. Compute $\int_0^1 \ln x dx$.

Solution: $\int \ln x dx = x \ln x - x + C$ (integration by parts). So we want to evaluate $x \ln x - x$ at $x = 1$ and $x = 0$. But $x \ln x$ is *not* defined at $x = 0$! This is where $\boxed{3}$(a) comes in. You are supposed to interpret $\int_0^1 \ln x dx$ as $\lim_{t\to 0} \int_t^1 \ln x dx$. So instead of $x \ln x - x|_{x=0}^1$, we need to do $\lim_{t\to 0} x \ln x - x|_{x=t}^1$ to get the final answer. The result is $\int_0^1 \ln x dx = \lim_{t\to 0} x \ln x - x|_{x=t}^1 = \lim_{t\to 0}(0 - 1) - t \ln t = -1$, because $\lim_{t\to 0} t \ln t = 0$.

In this example, we had a discontinuity at an edge of the region of integration. A more substantial modification is required when the integrand has a discontinuity inside the region of integration.

Exercise 5. Compute $\int_1^3 \frac{dx}{(x-2)^2}$

Solution: (Wrong solution) $\int_1^3 \frac{dx}{(x-2)^2} = -\frac{1}{x-2}\big|_{x=1}^3 = -2$.

This is wrong. In fact, the integral does not converge. The error in the computation above is in ignoring the discontinuity of $\frac{1}{(x-2)^2}$ at $x = 2$ (because the function is not defined there).

To get the correct answer, we need to use the definition of integrals whose integrands have discontinuities. To do so, we split the integral into two at the discontinuity, i.e. $\int_1^3 \frac{dx}{(x-2)^2} = \int_1^2 \frac{dx}{(x-2)^2} + \int_2^3 \frac{dx}{(x-2)^2}$. Then, evaluate the two integrals separately using $\boxed{3}$(a,b). $\int_1^2 \frac{dx}{(x-2)^2} = \lim_{t\to 2}\int_1^t \frac{dx}{(x-2)^2} = \lim_{t\to 2} -\frac{1}{x-2}\big|_{x=1}^t = \lim_{t\to 2} -\frac{1}{t-2} - 1 = \infty$. (Here, $\lim_{t\to 2}$ is to be intepreted as $\lim_{t\to 2^-}$, because you integrate from 1 to 2.) So the integral diverges, by $\boxed{3}$(c).

The last topic in this section is Comparison Theorem. Note that both $f(x)$ and $g(x)$ must be non-negative (i.e. $f(x), g(x) \geq 0$) in the region of integration. Example 9 is a good demonstration of this theorem, because e^{-x^2} is a function whose antiderivative we cannot find. Nevertheless, we can show that the integral converges thanks to the theorem.

Problems

1. $\int_{-\infty}^{\infty} xe^{-x^2} dx$
2. $\int_0^1 \ln x dx$
3. $\int_{-\infty}^{\infty} \cos \pi t dt$
4. $\int_1^{\infty} \frac{\ln x}{x} dx$
5. $\int_0^3 \frac{dx}{x-1}$
6. $\int_{-2}^3 \frac{dx}{x^4}$

7. $\int_{\pi/2}^{\pi} \csc x dx$

8. $\int_{-\infty}^{0} \frac{dx}{3-4x}$

9. Exercises 1, 2, 5-40, 49-54, 57-59 (p551-552)

Sections 11.1–7

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11.1: Sequences

A sequence is a discrete version of a function. The limit of a sequence is defined in essentially the same way. Most theorems that hold for limits of functions also hold for limits of sequences. One exception is l'Hopital's rule. However, there is a way to apply l'Hopital's rule indirectly to computing limits of sequences, as the following example demonstrates.

Exercise 1. Find $\lim_{n\to\infty} \frac{\ln(1/n)}{n}$.

Solution: You should first note that this limit is indeterminate. The idea here is to use Theorem $\boxed{3}$. Consider $f(x) = x \ln x$. The theorem tells you that $\lim_{n\to\infty} \frac{\ln(1/n)}{n} = \lim_{x\to 0} x \ln x$. You know from Section 6.8 that $\lim_{x\to 0} x \ln x = 0$, where

The intuitive definition of limit usually works, but when you are confused, use the formal definition $\boxed{2}$. I think of the definition in $\boxed{2}$ in this rephrased version: the limit of a_n is L (and denote $\lim_{n\to\infty} a_n = L$) if a_n is close to L for every large enough n.

Note that $\lim_{n\to\infty} a_n = \infty$ means something a bit different from the definition of limit when the limit is finite. See $\boxed{5}$. It says: for any positive number M, a_n is bigger than M when n is large enough.

Not every sequence is convergent. Take $a_n = \cos(n\pi)$ for example. $\lim_{n\to\infty} a_n$ does not exist because of the same reason $\lim_{x\to\infty} \cos(\pi x)$ does not converge.

The table in p717 lists algebraic properties of limit. You see the limit operation for sequences behave in the same way as that for functions. lim is a bit nicer than $\int$ in that $\lim(a_n \cdot b_n) = \lim a_n \cdot \lim b_n$ holds.

Theorem $\boxed{6}$ is an extremely useful one. It says: if the limit of the absolute values of terms is zero, then the original sequence also converges to zero.

Exercise 2. Show that $\lim_{n\to\infty} \frac{\sin n\pi}{n} = 0$.

<u>*Solution*</u>: We have $\left|\frac{\sin n\pi}{n}\right| \leq \frac{1}{n}$. So $\lim_{n\to\infty} \left|\frac{\sin n\pi}{n}\right| \leq \lim_{n\to\infty} \frac{1}{n} = 0$. Therefore, $\lim_{n\to\infty} \frac{\sin n\pi}{n} = 0$ by Theorem $\boxed{6}$.

Remark: Theorem $\boxed{6}$ and bounding absolute values of terms from above are commonly used together.

Another useful theorem is the Squeeze Theorem (p718, above $\boxed{6}$). (You must've seen this theorem used for functions.)

Exercise 3. Show that $\lim_{n\to\infty} \frac{1}{n} \sin(1/n) = 0$.

<u>*Solution*</u>: Since $-1 \leq \sin(1/n) \leq 1$, we have $-\frac{1}{n} \leq \frac{1}{n} \sin(1/n) \leq \frac{1}{n}$. Both $\frac{1}{n}$ and $-\frac{1}{n}$ goes to zero as $n \to \infty$. Then, by the squeeze theorem, $\lim_{n\to\infty} \frac{1}{n} \sin(1/n) = 0$.

Theorem $\boxed{12}$ gives an easy criterion for determining whether a increasing (or decreasing) sequence converges. Note, however, that it doesn't tell you the limiting value of a sequence. This theorem will be a useful tool in later sections when we discuss convergence of series.

Problems

Determine whether the sequence converges or diverges. If it converges, find the limit.

1. $a_n = \frac{1+n}{n^2}$
2. $a_n = (-1)^{n^2}$
3. $a_n = \frac{3^n}{1+2^n}$
4. $a_n = 1 + \frac{4^n}{5^n}$

5. $a_n = \frac{5^n}{4^n}$

6. $a_n = n \sin(1/n)$

7. $a_n = ne^{-n}$

8. $a_n = \sqrt{\frac{3+2n^2}{8n^2+n}}$

11.2: Series

You can think of a series as a sum of infinitely many numbers. You can also think of series as a discrete version of integration. In Section 7.8, we discussed what it means for an integral to converge or diverge. The rest of this chapter is devoted to discussing when convergence properties of series.

Usually, one starts with a series to define a sequence. Let $\{a_n\}$ be a sequence. Then the corresponding series is $\sum_{n=1}^{\infty} a_n$ (you don't have to start from $n = 1$; any number works). Let me start with a few examples before getting into the formal definition.

1. Consider the sequence $a_n = 1$. Then, the corresponding series is $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 1$. Intuitively, this series diverges to ∞.

2. Consider the sequence $a_n = \left(\frac{1}{2}\right)^n$. The corresponding series is $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$. By a geometric argument, one can show that $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$.

3. This is the last preliminary example and also my favorite one. This example illustrates a reason why we should care about convergence of a series. Let $a_n = (-1)^{n-1}$. Then, the corresponding series is $\sum_{n=1}^{\infty}(-1)^{n-1}$. Let's try to compute the value of this series with an elementary method. (Warning: whatever happens below is **wrong**.) First, write out terms of the series:

$$\sum_{n=1}^{\infty}(-1)^{n-1} = 1 + (-1) + 1 + (-1) + 1 + (-1) \cdots$$

Note that the pattern $1+(-1)$ repeats. If we group up these pattern and do additions within each patten, we get

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1} &= (1+(-1))+(1+(-1))+(1+(-1))\cdots \\ &= (0)+(0)+(0)\cdots\end{aligned}$$

So the answer is 0! (Reminder: this is wrong.)

Actually, we can group terms together in a different way. Leave the first 1 term, and group together the pattern $(-1)+1$:

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1} &= 1+((-1)+1)+((-1)+1)+((-1)+1)\cdots \\ &= 1+(0)+(0)+(0)\cdots \\ &= 1\end{aligned}$$

So we got 1 as an answer. Therefore,

$$0=\sum_{n=1}^{\infty}(-1)^{n-1}=1$$

So we have a proof that $0=1$? Something must have went wrong. In fact, $\sum_{n=1}^{\infty}(-1)^{n-1}$ is a *divergent* series. It means that we don't assign a value to $\sum_{n=1}^{\infty}(-1)^{n-1}$ at all.

Formally, a series is defined as the limit of "partial sums." In the notation of $\boxed{2}$, we say that a series converges if the limit of s_n exists. When the limit does not exists, we say that it diverges. As in the case of integrals, a series may "diverge to infinity," but we don't say that it "converges to infinity." Again, $\pm\infty$ is not considered a number.

Just like integrals, there aren't many series that we know how to evaluate. The geometric series ($\boxed{4}$) is one of the few series that we know the formula for. Note that the formula holds only if $|r|<1$. In other cases ($|r|>1$), the geometric series diverges.

Exercise 4. Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and its sum is 1. (This is Example 7.)

The Test for Divergence $\boxed{6}$ says that, if the tail of a sequence does not go to zero, then the series diverges. You must note that the converse is not true. There are sequences that go to zero, but the corresponding The following is an important example.

Exercise 5. Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Solution: This series is so important that it has a name. It's called the harmonic series. A proof that the harmonic series is divergent is worked out in Example 8. The integral test, which we discuss in 11.3, gives a nicer proof.

$\boxed{8}$ shows algebraic properties of $\sum_n$. Note that $\sum_n$ behaves like $\int$. In particular, you cannot distribute $\sum_n$ over a product, i.e. $\sum_n a_n b_n \neq \sum_n a_n \cdot \sum_n b_n$.

Problems

1. When is $\sum_{n=0}^{\infty} x^n$ convergent?
2. Compute $\sum_{n=0}^{\infty} x^n$ (assuming it's convergent).
3. Is $\sum_{n=1}^{\infty} \sin n$ convergent?
4. Is $\sum_{n=1}^{\infty} \ln n$ convergent?
5. Is $\sum_{n=1}^{\infty} \sqrt[n]{2}$ convergent?
6. Compute $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$.
7. Compute $\sum_{n=5}^{\infty} \frac{1}{3^n}$.
8. Compute $\sum_{n=1}^{\infty} \frac{7^{n+1}}{10^n}$
9. Compute $\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$.

11.3: The Integral Test & Estimates of Sums

You will see in 11.8–11.10 that every reasonably "nice" functions can be represented by series, and this representation is often useful in applications (such as physics). The main topic of our discussion until 11.7 will be convergence tests of series, which will be used in 11.8–11.11. So far, we know the convergence condition for the geometric series, and have the Divergence Test (p733). This section introduced the Integral Test, and more tests are to come in subsequent sections.

The Integral Test (p740) is what you want to consider using when the terms of a sequence are easy to integrate. You should read the "NOTE" right beneath the statement of the test. Also note that your sequence a_n needs be positive and decreasing, so you can't use the test for series like $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ or $\sum_{n=1}^{\infty} \sin n + 1$. Read the geometric explanations in p738-739, then take a look at Example 2. In particular, it gives a proof that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, which we left out in 11.2. Note the correspondence between $\boxed{1}$ and the result about $\int_1^{\infty} \frac{dx}{x^p}$ (p547).

When working with convergence of series, the following order of "rates of growth":

$$\ln n < \text{ polynomial } n^k < 2^n < e^n < n! \text{ as } n \to \infty$$

This says, essentially, that $\ln n$ goes to $+\infty$ much slower than any polynomial in n, and so on.

We are going to skip the part of this section about estimating the sum of a series. We will cover this material if it becomes necessary later. In practical applications, you have to approximate a series $\sum_{n=1}^{\infty} a_n$ by its finite sum (because it takes infinite time to add infinitely many numbers), when you can't find a formula for the series. In such a case, you want to know how close your finite sum, say $\sum_{n=1}^{10000} a_n$, is to the value of the series $\sum_{n=1}^{\infty} a_n$. Furthermore, as in Example 5(b), theoretical results can tell you how big your k needs to be, so that $\sum_{n=1}^{k} a_n$ is close enough to $\sum_{n=1}^{\infty} a_n$ for your purpose.

Problems

Determine whether the series converges.

1. $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n}$
2. $\sum_{n=1}^{\infty} \frac{1}{n^3}$
3. $\sum_{n=1}^{\infty} \frac{1}{3n+2}$
4. $\sum_{n=1}^{\infty} \frac{1}{n^\pi}$
5. $\sum_{n=5}^{\infty} n^{1-\sqrt{2}}$
6. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$
7. $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$
8. $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

11.4: The Comparison Tests

The Comparison Test for series works in almost exactly the same way as the Comparison Test for improper integrals (p549). Note that each term in both your sequence a_n and the sequence b_n to compare with need to be positive.

The Limit Comparison Test says when two sequences a_n and b_n have the same rate of growth ($= \lim_{n\to\infty} \frac{a_n}{b_n}$ = positive constant), then the corresponding series have the same convergence properties. It also requires that the terms are positive. Example 3 illustrates a case where the Limit Comparison Test works while the Comparison Test doesn't.

These are usually the candidates to compare your series against:

1. p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$
2. The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$
3. $\sum_{n=1}^{\infty} a_n$ such that $f(n) = a_n$ is easy to integrate

Problems

Determine whether the series converges.

1. $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n}$
2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4-1}}$
3. $\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n-2}$
4. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$
5. $\sum_{n=1}^{\infty} \frac{n+5}{n\sqrt{n}}$
6. $\sum_{n=1}^{\infty} \frac{n^2}{n^3-n+1}$
7. $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$

11.5: Alternating Series

We've seen that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. However, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent. When signs of the terms alternate

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \cdots,$$

$\frac{1}{2}$ cancels a bit of 1, $\frac{1}{4}$ cancels a bit of $\frac{1}{3}$, and so on. It turns out that the "cancellations" make the terms go to zero fast enough so that the series is convergent.

A generalization of this phenomenon is the Alternating Series Test. This test applies to series of the form $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$, where b_n is positive. Note: this test can only be used to test for convergence; i.e. you can't use this test to show a series is divergent.

Problems

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+4)}$

3. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$

4. $\sum_{n=1}^{\infty} (-1)^n \frac{4n-1}{3n+1}$

5. $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$

11.6: Absolute Convergence and the Ratio and Root Tests

This section introduces three convergence tests.

Given a sequence a_n, the series obtaitained from it (i.e. $\sum_{n=1}^{\infty} a_n$) is said to be **absolutely convergent**, if the series obtained from taking the absolute values of a_n is convergent. We use this concept as a convergence test: if a series is absolutely convergent, then it is convergent $\boxed{3}$. In many cases, it is easier to deal with $|a_n|$ than a_n itself. Example 3 is one such example. Note: this test can only be used to test for convergence; i.e. you can't use this test to show a series is divergent.

You should note that not every convergent sequence is absolutely convergent. The alternating harmonic series that we saw in the previous section is the archetypical conditionally convergent series.

Exercise 6. Show that the alternating harmonic series ($\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$) is conditionally convergent.

Solution: $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, because it's the harmonic series. On the other hand, we know that the alternating harmonic series is convergent by the Alternating Series Test.

The second test is the Ratio Test. We know that if $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$. The idea behind the Ratio Test is to see how fast a_n approaches 0. You should think of $\frac{a_{n+1}}{a_n}$ as the rate of change. Then, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ measures how fast the sequence decreases (or increases) when n is big (i.e. near the tail of the sequence). Intuitively, if the tail of a_n goes to 0 fast, then the series converges. This is exactly what the Ratio Test says: if

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ is strictly less than 1. On the other hand, the ratio is greater than 1, then the series is divergent.

The Ratio Test is gives you a convenient method to test for (absolute) convergence especially when the sequence involves powers and factorials (see Examples 4 and 5). However, there are somewhat surprising cases where the test is inconclusive.

Exercise 7. Show that we cannot conclude the convergence properties of $\sum_{n=1}^{\infty}\frac{1}{n^2}$ or $\sum_{n=1}^{\infty} n$.

Solution: For $a_n = \frac{1}{n^2}$,

$$\lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}\right| = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^2} = 1.$$

For $a_n = n$,

$$\lim_{n\to\infty}\frac{n+1}{n} = \lim_{n\to\infty} 1 + \frac{1}{n} = 1.$$

So the tests are inconclusive.

This is slightly disturbing. $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is obviously absolutely convergent, and $\sum_{n=1}^{\infty} n$ is obviously divergent! The moral of the story is that the Ratio Test is not very sentitive. However, because of its convenience, the Ratio Test is what I usually use first when testing for convergence of a series.

The third and last test introduced in this section is the Root Test. This test is a Ratio-Test-in-guise (read the note right below the statement of the Root Test). Instead of looking at the limit of $\left|\frac{a_{n+1}}{a_n}\right|$, the Root Test asks for the limit of $\sqrt[n]{|a_n|}$ (note the absolute value around a_n). Just like the Ratio Test, the Root Test is useful when a_n has the form $(\ Blah\ )^n$ (see Example 6). And just like the Ratio Test, you should watch out for the inconclusive case.

Problems

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

1. $\sum_{n=1}^{\infty} \frac{(-10)^n}{n^5}$

2. $\sum_{n=1}^{\infty} \frac{n!}{100^n}$

3. $\sum_{n=1}^{\infty} \frac{e^n}{2^{n+1}}$

4. $\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$

5. $\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{n^2}$

6. $\sum_{n=1}^{\infty} n \left(\frac{2}{3}\right)^n$

11.7: Strategy for Testing Series

This section is similar in spirit to 7.5. The only way to become good at solving problems is to solve problems, so go ahead and work on the end-of-section problems, sample exams, and extra problems on the course webpage (listed below).

You might find the flow chart on the department course webpage useful: `http://www.math.psu.edu/files/141seriesflowchart.pdf`

Problems

1. Do end-of-section problems.

2. `http://www.math.psu.edu/files/141series1.pdf`

3. `http://www.math.psu.edu/files/141series2.pdf`

4. `http://www.math.psu.edu/files/141series3.pdf`