Sections 11.8–11.11

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11.8: Power Series

In the second half of Chapter 11, we discuss one useful application of series. Before delve into the details, let me start with an overview of 11.8–11.11.

As I mentioned in passing a couple times, many functions can be represented as series. In fact, I used the fact that $\cos(x)$ has the "power series expansion" $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ in the demo of the alternating sereis estimation theorem. You can think of a power series as a polynomial with infinitely many terms. To make a power series, you start with a sequence c_n, which we refer to as the "coefficients." The power series corresponding to c_n is $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$.

For example, take $c_n = n$. The corresponding power series is $\sum_{n=0}^{\infty} nx^n = 0 + x + 2x^2 + 3x^3 + \cdots$. Just like for polynomials, we think of x as a variable. For $x = 0$, the power series evaluates to $\sum_{n=0}^{\infty} n \cdot 0^n = 0 + 0 + 2 \cdot 0^2 + 3 \cdot 0^3 + \cdots = 0$. For $x = 1$, the power series evaluates to $\sum_{n=0}^{\infty} n \cdot 1^n = \sum_{n=0}^{\infty} n$. Lets pause for a moment and think what's going on. When we pick a value for x, we get back a series. So we can consider a power series as a function: set $f(x) = \sum_{n=0}^{\infty} nx^n$. As we showed already, $f(0) = 0$. But what about when $x = 1$? We saw that $f(1) = \sum_{n=0}^{\infty} n$. By the Divergence Test, this series is divergent. Hence, $f(x)$ is *not defined* at $x = 1$. In summary, this is how we define $f(x)$ using the series $\sum_{n=0}^{\infty} nx^n$:

1. Pick $x = c$.

2. Consider the series $\sum_{n=0}^{\infty} nc^n$, and determine whether it converges.
3. If convergent, the function $f(x)$ is defined at $x = c$, and $f(c) = \sum_{n=0}^{\infty} nc^n$. Otherwise, $f(c)$ is undefined.

Try a few other values, and see if $f(x)$ is defined there.

In the previous paragraph, we started with a power series, and tried to define a function using it. The main topic of discussion in 11.9 is the opposite of this process. Given a function $f(x)$, in what domain of f can we represent it as a power series? Short answer: it depends on f. We will see examples later.

We learn applications of power series throughout the sections. Differentiation and integration become a simple task when you know the power series differentiation (Section 11.9). In 11.10 and 11.11, we see how the power series representation is used to estimate values of functions.

That's the overview of the rest of Chapter 11. Now moving onto 11.8. I said earlier that a power series is like a polynomial of an infinite degree. Just as you should draw a line between a series (which is like an infinite sum) and finite sum, you should make a clear distinction between a power series and polynomial.

With the aforementioned example $\sum_{n=0}^{\infty} nx^n$ and the discussion there in mind, read Example 1-3. Then figure out when $\sum_{n=0}^{\infty} nx^n$ is convergent.

Exercise 1. Show that $\sum_{n=0}^{\infty} nx^n$ is convergent when $|x| < 1$.

Solution: Let $a_n = nx^n$. We have

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)x^{n+1}}{nx^n}\right| = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) |x| = |x| .$$

When $|x| < 1$, we have $L < 1$ so the series is convergent. When $|x| > 1$, we have $L > 1$ so the series is divergent. When $|x| = 1$, the Ratio Test is inconclusive so we need to do extra work for $x = \pm 1$. When $x = \pm 1$, the series is divergent by the Divergence Test. In conclusion, $\sum_{n=0}^{\infty} nx^n$ is convergent when $|x| < 1$.

$\sum_{n=0}^{\infty} c_n(x-a)^n$ is called a power series "centered at a." To understand the terminology, think of $f(x) = 2x$, which can be thought of as a line through the origin with slope 2. If we replace x with $x - a$, where a is a positive number, we get $g(x) = f(x - a) = 2(x - a)$. Then, $g(x)$ is a line through $x = a$ with slope 2. So the operation of "replacing x with $x - a$" shifts a function to the positive x-direction by a.

Note that in Examples 1-3, you only used the Ratio Test to determine when the series is convergent. This is not a coincidence. In fact, when you test for convergent properties of a power series, the Ratio Test gets the job done in most cases. This is, in part, a consequence of Theorem 3, which says that power series have only three types of convergence. We will use the terms "radius of convergence" and "interval of convergence" frequently, so make sure you understand what they mean.

Problems

For what values of x does the series converge?

1. $\sum_{n=0}^{\infty} nx^n$
2. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n}$
3. $\sum_{n=0}^{\infty} \frac{(-3)^n}{n\sqrt{n}} x^n$
4. $\sum_{n=0}^{\infty} n^n x^n$
5. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$
6. $\sum_{n=0}^{\infty} \frac{n}{4^n} (x+1)^n$

11.9: Representations of Functions as Power Series

(When we say "expand $f(x)$ around $x = a$," it means "get a power series representation of $f(x)$ centered at $x = a$.")

Theorem 2 says that the rules of differenciation and integration for polynomials (which have finite degrees) hold for power series, as well. In other words, you can treat power series like polynomials when differentiating and integrating.

The power series representation in $\boxed{1}$ was proved in an earlier section. This formula can be used to find power series representations of some rational functions (see Examples 1-3). Note that $\frac{1}{1-x}$ is defined everywhere except for $x = 1$, but its power series representation is defined *only for* $|x| < 1$. Pick $x = -1$, for example. The function $f(x) = \frac{1}{1-x}$ evaluates to $f(-1) = \frac{-1}{2}$, but $sum_{n=0}^{\infty}(-1)^n$ doesn't have a value, because it diverges.

Let's play around with this function a little bit more. When we expand $f(x) = \frac{1}{1-x}$ around $x = 0$, the power series didn't carry any information of f at $x = -1$. Suppose that, for whatever reason, we are interested in knowing how f behaves near $x = -1$. To do so, we expand f at $x = -1$.

Exercise 2. Find the power series expansion of $\frac{1}{1-x}$ at $x = -1$.

Solution:

$$\frac{1}{1-x} = \frac{1}{2-(x+1)} = \frac{1}{2} \cdot \frac{1}{1-\frac{x+1}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x+1}{2}\right)^n.$$

This series converges when $\left|\frac{x+1}{2}\right| < 1 \iff |x+1| < 2 \iff -3 < x < 1$. Note that the interval of convergence for this expansion is larger than the expansion at $x = 0$. The expansion at $x = -1$ agrees with the expansion at $x = 0$, i.e. $\sum_{n=0}^{\infty}\left(\frac{x+1}{2}\right)^n = \sum_{n=0}^{\infty} x^n$, when $|x| < 1$.

Combining Theorem $\boxed{2}$ and $\boxed{1}$, we can find power series representations for a wider variety of rational functions. Read Examples 5-8.

Problems

Find a power series representation of the function and determine the interval of convergence.

1. $\frac{1}{x+1}$
2. $\frac{1}{x+2}$
3. $\frac{x^2}{x+2}$
4. $\frac{x}{2x^2+1}$

5. $\frac{5}{1-4x^2}$

6. $\frac{3}{x^2-x-2}$ (use partial fraction)

7. $\ln(3-x)$

8. $\ln(x^2+4)$

9. $x^2\tan^{-1}(x^3)$

10. $\frac{x}{(1+4x)^2}$

11. $\frac{(x^2+x)}{(1-x)^3}$

11.10: Taylor and Maclaurin Series

In 11.8, we were given a power series, and asked where it converged. In 11.9, we were given a rational function (and its derivatives and antiderivatives), and asked where it had a power series representation. This section is a continuation of 11.9. Note that, in 11.9, we heavily depended on the power series expansion of $\frac{1}{1-x}$ to find power series. Thus, so far, we only know how to find power series expansion for rational functions and functions related to rational functions. We expand our scope in this section through studying Taylor series.

Theorem $\boxed{5}$ gives you a formula to find the cofficient of a power series for $f(x)$. It's really important to note that this formula holds **given the power series converges at** $x = a$. The series in $\boxed{6}$ is called the Taylor series (or Taylor expansion) of f. As the statement in $\boxed{5}$ says, the power series expansion of f at $x = a$ exists, then it is given by the Taylor series. So now the question is this: given a function f and a point $x = a$, how do we determine whether a power series expansion of f at a exists?

To answer that question, we go back to the definition of series. Try to understand the discussion in p779 and Theorem $\boxed{8}$ first, then proceed to my version of the same discussion. By Theorem $\boxed{5}$, we know that if the power series expansion of f at $x = a$ exists at all, then expansion has to look like this:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

This tells you that, if $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ converges, then the series equals $f(x)$ (in an appropriate domain). This motivates you to consider $T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$, which is the k-th partial sum of the Taylor series. In the book, $T_k(x)$ is called the k-th degree Taylor polynomial of f at a. $\lim_{k\to\infty} T_k(x)$ is the only candidate for the value of $f(x)$. If $\lim_{k\to\infty} T_k(x)$ converges (which is equivalent to say $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ converges), then its limit coincides with $f(x)$. If $\lim_{k\to\infty} T_k(x)$ diverges, then we conclude that we can't assign a value to $f(x)$ using this power series, because there's no other candidate. Now, let's talk about the remainder R_n, which is defined as $R_n(x) = f(x) - T_n(x)$. In other words, R_n is the difference between the function and the n-th degree Taylor polynomial. Intuitively, we have $T_n(x) \to f(x)$ as $n \to \infty$ if and only if $R_n \to 0$. This is what Theorem $\boxed{8}$ says.

It's necessary to consider R_n, because we want to make sure not only that $\lim_{n\to\infty} T_n(x)$ **converges, but also the limit value coincides with** $f(x)$**.** For example, Example 2 tells you that e^x has the Maclaurin series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x. You could easily show that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x by the Ratio Test, but **this doesn't mean that the series converges to** e^x**!** In most cases, the limit of R_n is difficult to determine. For example, R_n for e^x is given by $R_k(x) = e^x - \sum_{n=0}^{k} \frac{x^n}{n!}$. Without the knowledge that $\lim_{k\to\infty} \sum_{n=0}^{k} \frac{x^n}{n!} = e^x$, it's hard to figure out what the limit is going to be. This is why Theorem $\boxed{9}$ is useful.

Examples 2-5 discuss how to find power series expansions of e^x, $\sin x$, and $\cos x$. These serve as illustration of Theorem $\boxed{8}$ and $\boxed{9}$. Also, you should know the Maclaurin series for these functions.

The table in p786 lists the Maclaurin series that you should know.

Examples 10-13 are some applications of the techniques that we've learned so far. I think all of them are neat.

Problems

1.

11.11: Applications of Taylor Polynomials

Problems

1.

Sections 10.1–4

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10.1: Curves Defined by Parametric Equations

There are common geometric objects that are not possible to represent mathematically as the graph of a function. The circle is an archetypical example. The graph of $f(x) = \sqrt{1-x^2}$ in the region $x \in [-1, 1]$ is the upper half of the circle centered at the origin of radius 1. In order to describe the lower half, we need another function $g(x) = -\sqrt{1-x^2}$. That we need two functions to describe is not only cumbersome, but also brings issues, for example, when we want to find the equation of the tangent line at the points where $f(x)$ and $g(x)$ intersect, namely, $(1, 0)$ and $(-1, 0)$. The main issue in using the graph of a function to represent a geometric object is that a you can assign only one y-value to each x-value, because otherwise a property of function is violated. One way to get around this limitation is to describe the circle by a Cartesian equation: $x^2 + y^2 = 1$. Another way is to use the "parametric equation." Let $f(t) = \cos t$ and $g(t) = \sin t$. Then, define a (vector-valued) function $(f(t), g(t))$. This is a function that take a value for t, and returns a point in the plane, where $f(t)$ is the x-coordinate, and $g(t)$ is the y-coordinate. We say that the parametric equation $x = f(t)$, $y = g(t)$ $(0 \leq t \leq 2\pi)$ represents the circle.

You should think of the parametric equation $(f(t), g(t))$ as a function whose domain is *time* t. Suppose that the parametric equation gives a curve $\mathcal{C}$. Imagine that you are a point on $\mathcal{C}$, at time $t = 0$, you are sitting at the point $(f(0), g(0))$ on the curve, and you move forward as t grows.

Examples 2-3 show how to go from a parametric equation to a Cartesian equation. It's also possible to go from a Cartesian equation to a parametric equation (Example 6).

The parametric equation has the advantage over a Cartesian equation that it is a function. This allows us to calculate the length of a curve using calculus, which is the topic of discussion in the next section.

Problems

Eliminate the parameter to find a Cartesian equation of the curve

1. $x = \cos\theta$, $y = \sin\theta$ $(-\pi \leq \theta \leq \pi)$
2. $x = 2\cos\theta$, $y = 2\sin\theta$ $(0 \leq \theta \leq 2\pi)$
3. $x = 3 - 4t$, $y = 2 - 3t$
4. $x = 1 - 2t$, $y = \frac{1}{2}t - 1$ $(-2 \leq t \leq 4)$
5. $x = \sqrt{t}$, $y = 1 - t$
6. $x = \frac{1}{2}\cos\theta$, $y = 2\sin\theta$ $(0 \leq \theta \leq 2\pi)$
7. $x = \sin t$, $y = \csc t$
8. $x = e^{2t}$, $y = e^{2t}$

10.2: Calculus with Parametric Curves

If we are given a curve *which is the graph of a function*, we know how to compute the tangent line of a point, the area under the curve, and the length of the curve. However, as discussed in the previous section, not every geometric object can be described using functions easily. This section discusses how to compute these quantities for a curve described by a parametric equation.

Given a curve and a point, we can find the tangent line at the point. The formula is $\boxed{1}$. You also find the formula for $\dfrac{d^2y}{dx^2}$, which you use to find the curvature of a curve, at the bottom of the same page. When a curve self-intersects, you may find more than one tangent lines. Read Examples 1 and 2.

The area under the curve $x = f(t)$, $y = g(t)$ is given by the formula in p671. (Note: the author uses a change of variable from x to t via $x = f(t)$ to derive the formula.)

Exercise 1. Find the area of the upper half unit disk.

Solution: The parametric equation for the upper half circle is $x = \cos\theta$, $y = \sin\theta$ for $0 \leq \theta\pi$. The area under it is given by

$$\int_\pi^0 \sin\theta(\cos\theta)' d\theta = -\int_\pi^0 \sin^2\theta d\theta = -\int_\pi^0 \frac{1-\cos 2\theta}{2} d\theta = -\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)_\pi^0 = \frac{\pi}{2}.$$

The formula in Theorem $\boxed{5}$ gives the length of a curve between time $t = \alpha$ and β. Example 5 computes the length of a curve represented as a parametric curve, which is not obvious how to represent using a Cartesian equation.

An application of $\boxed{5}$ is the surface area of an object obtained by rotating a curve around an axis. Read Example $\boxed{6}$.

Problems

Find an equation of the tangent(s) to the curve at the point corresponding to the given value of the parameter.

1. $x = t\cos t$, $y = t\sin t$, $t = \pi$
2. $x = 6\sin t$, $y = t^2 + t$, $t = 0$

Find dy/dx and d^2y/dx^2.

1. $x = t^3 + 1$, $y = t^2 - t$.
2. $x = \cos 2t$, $y = \cos t$.

1. Find the points on the curve $x = t^3 - 3t$, $y = t^2 - 3$ where the tangent is horizontal or vertical.

2. Find the area enclosed by the curve $x = \sqrt{t}$, $y = t^2 - 2t$ and the x-axis.

3. Find the area enclosed by the curve $x = 1 + e^t$, $y = t - t^2$ and the x-axis.

Set up an integral that represents the length of the curve. Don't evaluate it.

1. $x = t^2 - t$, $y = t^4$, $1 \leq t \leq 4$

2. $x = t + \sqrt{t}$, $y = t - \sqrt{t}$, $0 \leq t \leq 1$

Find the length of the curve.

1. $x = e^t + e^{-t}$, $y = 5 - 2t$, $0 \leq t \leq 3$

2. $x = 3\cos t - \cos 3t$, $y = 3\sin t - \sin 3t$, $0 \leq t \leq \pi$

3. $x = \cos t + \ln(\tan \frac{t}{2})$, $y = \sin t$, $\frac{\pi}{4} \leq t \leq \frac{3\pi}{4}$

Find the area of the surface obtained by rotating the curve about the x-axis.

1. $x = 3t - t^3$, $y = 3t^2$, $0 \leq t \leq 1$

10.3: Polar Coordinates

$\boxed{1}$ shows the familiar rule for going between a Cartesian equation and a polar equation. The equations in $\boxed{2}$ ($r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$) is a consequence of $\boxed{1}$. $\boxed{2}$ comes in handy when converting a polar equation to a Cartesian equation (see Example 6b). On the other hand, to go from a Cartesian equation to a polar equation, you use $\boxed{1}$. Since there are not many illustrative examples in this section, try a couple end-of-section problems (15-26) to get used to using $\boxed{1}$ and $\boxed{2}$. As a convention, a polar equation has the form $r = f(\theta)$, i.e. r equals a function in θ. That is, you may need to manipulate a equation to isolate r from θ.

For now, you can skip the paragraph titled "Symmetry" (p683). This is not something absolutely necessary, although you might want to come back to it when sketching curves like $\cos 2\theta$.

Recall that, when you did u-substitutions in integrals, you had to watch out what happens to dx. Similarly, formulas involving derivatives require modifications when changing variables. Recall the formula from the previous section

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$

To get the formula for polar coordinates, we want to use the following change of variables $x = r\cos\theta$, $y = r\sin\theta$. Differentiate both sides of $x = r\cos\theta$ with respect to θ to get $\dfrac{dx}{d\theta} = \dfrac{dr}{d\theta}\sin\theta - r\sin\theta$ (note the product rule). Similarly, we get $\dfrac{dy}{d\theta} = \dfrac{dr}{d\theta}\cos\theta + r\cos\theta$. Hence, the formula for $\frac{dy}{dx}$ of polar curves is the following

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta + r\sin\theta}.$$

You should become familiar with the examples described in the section. Among other things, you should learn to sketch them. You will encounter these polar curves frequently, and on the other hand, you won't see polar curves that are very different from these. I will demonstrate how to sketch as many examples as possible in class.

Problems

Find a polar equation for the curve represented by the given Cartesian equation.

1. $y = -1$
2. $4y^2 = x$

Find a Cartesian equation for the curve.

1. $r = 4\sec\theta$

2. $r = \cos\theta$

3. $\theta = \dfrac{\pi}{3}$

4. $r = \tan\theta\sec\theta$

Find a polar equation for the curve represented by the given Cartesian equation.

1. $y = x$

2. $4y^2 = x$

3. $xy = 4$

4. $y = 1 + 3x$

Sketch the graph of the polar curves.

1. $r = 1 + \sin\theta$

2. $r = \cos 2\theta$

3. $r = \sin\theta$

4. $r = 1 + \sin\theta$

5. $r = 1 - \cos\theta$

6. $r = 1 + 2\cos\theta$

7. $r = \ln\theta$

8. $r = \cos 5\theta$

9. $r = 2 + \sin\theta$

Find the slope of the tangent line to the given polar curve at the specified point.

1. $r = 2 - \sin\theta$, $\theta = \frac{\pi}{3}$

2. $r = \cos(\theta/3)$, $\theta = \pi$

10.4: Areas and Lengths in Polar Coordinates

Recall that the area of a sector is given by $\frac{1}{2}r^2\theta$. It is possible to generalize this formula. The area of the region between the origin and a polar curve $r = f(\theta)$ from $\theta = a$ to $\theta = b$ (I will explain what this wording really means in class) is given by

$$\int_a^b \frac{1}{2}(f(\theta))^2 d\theta.$$

Note that, when $f(\theta)$ is a constant function, we get the area of a sector. Read Example 1. Example 2 gives an example that shows how to compute the area between two curves.

Read the "CAUTION" in p691. The main point is that solving equations doesn't necessarily give you all points of intersections.

$\boxed{5}$ is the polar curve version of the arc length formula. Compare it with $\boxed{5}$ in p673.

Problems

Sketch the curve and find the area it encloses.

1. $r = 1 - \sin\theta$

2. $r = 4 + 3\sin\theta$

Find the area of the region enclosed by one loop of the curve.

1. $r^2 = \sin 2\theta$

Find the area of the region that lies inside the first curve and outside the second curve.

1. $r = 1 - \sin\theta$, $r = 1$

2. $r = 2 + \sin\theta$, $r = 3\sin\theta$

3. $r = 3\sin\theta$, $r = 2 - \sin\theta$

Find the area of the region that lies inside both curves.

1. $r = 1 + \cos\theta$, $r = 1 - \cos\theta$

2. $r = 3 + 2\cos\theta$, $r = 3 + 2\sin\theta$

Find the length of the polar curve.

1. $r = 5^{\theta}$, $0 \leq \theta \leq \pi$

2. $r = 2(1 + \cos\theta)$, $0 \leq \theta \leq 2\pi$